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A finite temperature generalization of Zamolodchikov's C-theorem

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Abstract: We prove a C-theorem within the framework of two dimensional quantum field theories at finite temperature. There exists a function $C(g_1, g_2, ..., g_n)$ of coupling constants which is non-increasing along renormalization group trajectories and non-decreasing along temperature trajectory and stationary only at the fixed points. The connection between the C-theorem at zero temperature and the C-theorem at finite temperature is discussed. We also consider the thermodynamical aspects of the C-theorem. If we define the C-function in an arbitrary number of dimensions in analogy to the two dimensional case, we can show that its behavior is not universal. The phase transitions destroy the monotonic properties of the C-function. The proof of the C-theorem is also presented within the framework of the Källen-Lehmann spectral representation at finite temperature.

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1 Introduction

The formalism of standard QFT is suitable for describing observables measured in empty space-time, such as particle interactions in an accelerator. In some interesting physical situations (for example, the early stages of the universe) the environment has a non-negligible matter and radiation density, making the assumptions of standard QFT impracticable. For that reason, we have to look for a new formalism which is closer to thermodynamics. Starting with the basic principles of equilibrium statistical mechanics, the formalism of Feynman's functional integral is used to go from an expression for the time translation operator $exp(-iHt)$ to the partitition function of the grand canonical ensemble $Tr(exp(-\beta H))$ by means of analytical continuation to imaginary time. This formalism is called QFT at finite temperature. On the other hand, the methods of field theory at finite temperature are useful and important for the theory of phase transitions which in turn is connected to spontaneous symmetry breaking in QFT at zero temperature. In this sense QFT at finite temperature is important for undestanding different phases of zero temperature QFT. In this work we study some general properties of an unitary 2D QFT at finite temperature. We also discuss the relation between these results and thermodynamics.

Two dimensional field theories have remarkable properties. One of these properties is that the motion in the space of dimensionless coupling constants under influence of RG group is an "irreversible" process. This is the content of the Zamolodchikov's C-theorem which establishes the existence of a dimensionless function of the coupling constants with monotonic properties along RG trajectories [1]. We will prove that it is possible to generalize this theorem within the framework of finite temperature 2D QFT. Namely: there exists a function of coupling constants with monotonic properties along RG trajectories and along temperature trajectory. In 2D QFT the RG and temperature flows have similar behavior in the context of the C-function

$$\Lambda \frac{\partial C}{\partial \Lambda} = -T \frac{\partial C}{\partial T} \leq 0 \tag{1}$$

where Λ is the mass scale of the theory and T denotes temperature. This property provides us with a useful tool for studying phases of 2D QFT. We will show that in 2D QFT existence of a C-function is connected to the triviality of one dimensional statistical mechanics. The finite temperature formu-

lation of C-theorem gives us new insights into the properties of 2D QFT and explains the difficulties that appear in trying to generalize Zamolodchikov's theorem to more than two dimensions.

In section 2 we review the different proofs of Zamolodchikov's original theorem. In particular we remind the reader of the proof where the spectral representation for the two-point function of the stress tensor is used [2]. This proof presents a nice physical picture of the RG flow.

In section 3 we explain what we mean by 2D QFT at finite temperature and also introduce some notation.

In section 4 we prove the C-theorem using the finite size approach advocated by Ludwig and Cardy [3]. We note that there exists a non-trivial relation between the zero temperature and finite temperature C-theorems. In the finite temperature formulation of the C-theorem the theory becomes conformally invariant at zero temperature. This IR critical point describes the well-known phase transition at zero temperature for one dimensional statistical system.

In section 5 we disscus a thermodynamical proof of the C-theorem. The monotonic properties of the C-function require an additional thermodynamical condition. The standard thermodynamical axioms do not imply this condition and this shows that the C-theorem is not a universal property of the QFT in arbitrary number of dimensions. The phase transitions destroy the monotonic behavior of the C-function.

In section 6 we generalize the Källen-Lehmann spectral representation to the two-point correlator at finite temperature.

In section 7 we use the this representation of two-point function of the stress tensor to prove the finite temperature C-theorem.

2 The C-theorem at zero temperature

Irreversibility of the RG flow is proved in two dimensions by Zamolodchikov [1]. It says that there exists a function $C(g_1, g_2, ..., g_n)$, which is monotonically decreasing along the RG flow

$$\Lambda \frac{dC}{d\Lambda} = -\beta_i(g) \frac{\partial}{\partial g_i} C(g) \leq 0 \tag{2}$$

and stationary for conformally invariant theories

$$\beta_i(g) = 0 \quad \Longleftrightarrow \quad \frac{\partial C}{\partial g_i} = 0 \tag{3}$$

where it takes the value of the Virasoro central charge. Zamolodchikov's proof is very simple and uses the conditions of renormalizability, positivity, the translational and rotational symmetries (the Poincaré symmetry), and certain special properties of a 2D conformal field theory.

Let us sketch it. Consider the stress tensor $\mathcal{T}_{\mu\nu}$ and introduce the notation $\Theta \equiv \mathcal{T}^{\mu}_{\mu}$ and $\mathcal{T} \equiv \mathcal{T}_{zz}$, where $z = x_1 + ix_2$ are complex coordinates. The correlators of the stress tensors have no anomalous dimensions so they can be parametrized as follows,

$$< \mathcal{T}(z,\bar{z})\mathcal{T}(0,0) >= \frac{F(z\bar{z}\Lambda)}{z^4} \qquad < \Theta(z,\bar{z})\mathcal{T}(0,0) >= \frac{G(z\bar{z}\Lambda)}{z^3\bar{z}}$$

$$< \Theta(z,\bar{z})\Theta(0,0) >= \frac{H(z\bar{z}\Lambda)}{z^2\bar{z}^2} \tag{4}$$

The Poincaré invariance[1] requires conservation of the stress tensor, which gives different relations between the scalar functions F, G, H. Let us introduce the quantity

$$C = 2(F - \frac{1}{2}G - \frac{3}{16}H) \tag{5}$$

which satisfies

$$\dot{C} = -\frac{3}{4}H, \qquad \dot{C} \equiv z\bar{z}\frac{dC}{d(z\bar{z})} \tag{6}$$

Unitarity of the theory implies that H is positive definite. Then C reduces to the central charge at the fixed point ($\Theta = 0$). Introducing the renormalization point we obtain the C-function as a function of coupling constants

$$C(g_1, g_2, ..., g_n) = C|_{|z|=1} \tag{7}$$

An alternative proof of the C-theorem was given by Friedan [4], using the Källen-Lehmann spectral representation of the correlator of two stress

[1] By Poincaré invariance we mean the rotational and translational invariance in euclidian QFT.

tensors

$$< \mathcal{T}_{\mu\nu}(x)\mathcal{T}_{\rho\sigma}(0) >= \frac{\pi}{3}\int\limits_0^\infty d\mu c(\mu)\int \frac{d^2p}{(2\pi)^2}e^{ipx}\frac{(g_{\mu\nu}p^2 - p_\mu p_\nu)(g_{\rho\sigma}p^2 - p_\rho p_\sigma)}{p^2+\mu^2} \tag{8}$$

In two dimensions there is only one Lorentz invariant object with four indices which can be constructed from p_μ and which is compatible with conservation of the stress tensor. Consequently in the spectral representation we have only one unknown scalar function of the intermediate mass scale μ namely the spectral density $c(\mu)$. This density also depends on the mass scale Λ of the theory, as well as on the dimensionless coupling constants. The proof of the C-theorem goes on by establishing the properties of $c(\mu)$ [2]. Unitarity of the theory gives $c(\mu) \geq 0$. The form of $c(\mu)$ in a scale invariant theory is completly fixed by its dimensionality. In two dimensions the scale invariance is equivalent to conformal invariance so this argument provides a good definition of the spectral density. Therefore in theories with no scale invariance the general form of $c(\mu)$ is

$$c(\mu) = c_0\delta(\mu) + c_1(\mu,\Lambda) \tag{9}$$

where c_0 is the central charge of the Virasoro algebra and c_1 describes the behavior away from the critical point $\mu = 0$ and depends on the mass scale Λ of the theory. Since the correlators of the stress tensor cannot develop anomalous dimensions, the behavior of $c(\mu)d\mu$ under the RG flow is simply given by dimensional analysis

$$c_\lambda(\mu,\Lambda)d\mu = c(\lambda\mu,\Lambda)\lambda d\mu = c(\mu,\Lambda/\lambda)d\mu \tag{10}$$

where we disregard the $\delta(\mu)$ term. For further detailes of the proof we refer the reader to [2].

Let us briefly discuss the connection between Zamolodchikov's and Friedan's proofs. Zamolodchikov's C-function can be obtained by integrating the density $c(\mu)$ with smearing functions f with the following properties: $f > 0$, $f(0) = 1$, $f(\mu)$ decreases exponentially as $\mu \to \infty$ and $\mu df/d\mu \leq 0$

$$C(g(\Lambda)) = \int d\mu c(\mu)f(\mu) = \int d\mu c_1(\mu,\Lambda)f(\mu) + c_0 \tag{11}$$

Then the derivative along the RG flow is

$$-\beta_i(g)\frac{\partial}{\partial g_i}C = \Lambda\frac{\partial}{\partial\Lambda}C = \int d\mu c_1(\mu,\Lambda)\mu\frac{d}{d\mu}f(\mu) \leq 0 \tag{12}$$

where we have used (10) and integrated by parts. Zamolodchikov's choice corresponds to

$$\mu\frac{df}{d\mu} = -\frac{\pi}{2}\mu^4 G(|x| = 1, \mu) \tag{13}$$

where G is the propogator of a free scalar particle. In the next sections we will use the ideas of these poofs.

3 Finite temperature 2D QFT

We will consider the grand canonical ensemble of particles which are described by 2D QFT. In the grand canonical ensemble, the isolated system can exchange particles and energy with a reservoir. In this ensemble, the temperature T, the volume V and the chemical potential μ_i are fixed variabeles. The grand canonical partitition function is

$$Z = Tre^{[-\beta(H-\mu_i N_i)]} \qquad \beta = \frac{1}{T} \tag{14}$$

where H is the Hamiltonian of 2D QFT and N_i are a set of conserved number operators. In our further considerations we study the theory without conserved charges (if the system admits some conserved charge, then we have to make the replacement

$$H(\pi, \phi) \rightarrow \tilde{H}(\pi, \phi) = H(\pi, \phi) - \mu N(\pi, \phi) \tag{15}$$

where ϕ is the field and π is the conjugate momentum). Using the methods of functional integrals [5] we can rewrite (14) as

$$Z = N' \int\limits_{\substack{periodic \\ antiperiodic}} D\phi \;\; exp(\int\limits_0^\beta dx_1 \int dx_2 \;\; \mathcal{L}) \tag{16}$$

where $\mathcal{L}$ is the lagrangian density for 2D QFT and we use the imaginary-time formalism[2]. We integrate over the space of functions with periodic

[2]Note that we use the definitions and notation of Kapusta [5]. In another review [6] of the finite temperature field theory the partitition function is defined as $\int D\phi \; exp(-\int_0^\beta dx_1 \int dx_2 \mathcal{L})$. This definition does not change the general argument presented here, because the monotonic property does not depend on the sign. We can redefine the C-function and obtain the same results.

(antiperiodic) properties

$$\phi(\beta, x_2) = \pm\phi(0, x_2) \tag{17}$$

which are defined on the strip $\mathbf{R} \times [0, \beta]$. The sign depends on the statistics of the fields considered, "+" for bosons and "−" for fermions. The normalization constant N' is irrelevant for our argument.

Formally we can consider the finite temperature 2D QFT as a QFT defined on $\mathbf{R} \times \mathbf{S^1}$. More exactly we may generalize the periodic (antiperiodic) condition (18) to

$$\phi(x_1 + k\beta, x_2) = \pm\phi(x_1, x_2) \qquad k \in \mathbf{Z} \tag{18}$$

and consider QFT on $\mathbf{R^2}/\mathbf{\Lambda}$ where $\mathbf{\Lambda}$ is the lattice $\beta\mathbf{Z}$.

For zero temperature 2D QFT we have the Poincaré symmetry $\mathbf{P} = \mathbf{SL(2,R)} \triangleright \mathbf{E_2}$ which is a semidirect product of the Lorentz group $\mathbf{SL(2,R)}$ and the abelian group of translations $\mathbf{E_2}$. Irreducible representations of the Poincaré group are labelled by the eigenvalues of Casimir operators, the spin and the mass. In the finite temperature case, QFT is invariant under the group $\mathbf{P}/\mathbf{\Lambda}$ and the representations of this group can be constructed in the same way as for the standard Poincaré group. We have the same Casimir operators and the time-like component of the momentum is quantized in units of $2\pi T$. In other words the time-like component of the momentum is an element of the dual lattice $\mathbf{\Lambda}^*$.

We see that formally finite temperature 2D QFT can be treated as ordinary QFT on a cylinder $\mathbf{R} \times \mathbf{S^1}$. The finite temperature 2D QFT can be interpreted as an infinite one dimensional statistical system which has an infinite dimensional phase space and can be obtained from a finite statistical system by taking the thermodynamical limit

$$\lim_{\substack{N \to \infty \\ \frac{N}{V} = const.}} \int e^{-\beta H} \prod_{i=1}^{N} dq_i \prod_{j=1}^{N} dp_j = \int D\phi \; D\pi \; e^{-\beta H(\phi,\pi)} \tag{19}$$

where the number of particles N goes to infinity but the density of particles N/V is finite. We will show how the properties of finite temperature 2D QFT's are connected with the properties of infinite one dimensional statistical systems.

We finally define the free energy F of the grand canonical ensemble as

$$F \equiv -\log Z \tag{20}$$

and the free energy density $\mathcal{F}$ as

$$\mathcal{F} \equiv -\frac{1}{V} \log Z \tag{21}$$

where V is the volume of the strip $\mathbf{R} \times [0, \beta]$.

4 The C-theorem at finite temperature

We are going to use the finite size approach [3]. This is widely used for solving zero temperature problems, but here we would like to draw the readers attention to the application of these ideas to finite temperature QFT.

Let us introduce the function c

$$c(\lambda, T, a) = \frac{6}{\pi T^2}[\mathcal{F}(\lambda, 0, a) - \mathcal{F}(\lambda, T, a)] \tag{22}$$

where λ is a set of the coupling constants and a is an UV cut-off. The function c does not need any IR cut-off. The finite-size scaling methods have an infrared cut-off automatically. The UV divergencies of (22) are not particular to this construction since UV regularization and renormalization must be introduced at zero temperature, and are then unmodified at finite temperature.

Let us establish the properties of the function c. By the definitions (20) and (22) c is a dimensionless function. Consider the following infinitestimal transformation

$$x_1 \to x_1 + \xi_1 x_1, \qquad x_2 \to x_2 + \xi_2 x_2, \qquad |\xi_{1,2}| \ll 1 \tag{23}$$

where x_1 and x_2 denote the longitudinal and transversal coordinates on the strip $\mathbf{R} \times [0, \beta]$ which we denote as $\mathtt{S}$. The change in the free energy can be represented in terms of the stress tensor

$$\delta F = - < \delta S > = \frac{1}{2\pi} \int_{\mathtt{S}} d^2x < \mathcal{T}_{11} > \xi_1 + \frac{1}{2\pi} \int_{\mathtt{S}} d^2x < \mathcal{T}_{22} > \xi_2. \tag{24}$$

S denotes the action of our theory. On the other hand the change in F given by (24) is due to its scale dependence on the inverse temperature β and the UV cut-off (length). So we can read off

$$\beta\frac{\partial F}{\partial \beta} = \frac{1}{2\pi}\int\limits_{\mathtt{S}} d^2x < \mathcal{T}_{11} >= \frac{1}{2\pi}\int\limits_0^\beta dx_1 < H > \tag{25}$$

where H is the Hamiltonian of the system

$$H = \int\limits_{-\infty}^{\infty} dx_2 \mathcal{H} = \int\limits_{-\infty}^{\infty} dx_2 \mathcal{T}_{11}(x_1, x_2), \tag{26}$$

and $\mathcal{H}$ is the Hamiltonian density. In an unitary theory we have to define the ground state energy as a positive function of the parameters [7]. This assumption implies

$$T\frac{\partial F}{\partial T} = -\beta\frac{\partial F}{\partial \beta} \leq 0 \quad \Longleftrightarrow \quad c(\lambda, T, a)V \geq 0 \tag{27}$$

We have thus shown that the function c is positive definite as a consequence of the positivity of the energy of the ground state.

The free energy density $\mathcal{F}$ (λ, T, a) may be written as

$$\mathcal{F}(\lambda, T, a) = -\frac{1}{V}\log[\int D\phi \; exp(S_0 + \lambda_i \mu^{d_i}\int\limits_{\mathtt{S}} d^2x O^i(x))] \tag{28}$$

where the full action is represented as a deformation of the conformal theory S_0 by operators $O^i(x)$ of mass dimension $2 - d_i$. From (28) we may read off the expressions

$$\frac{\partial c}{\partial \lambda_i} = \frac{6\mu^{d_i}}{\pi T^2 V}[\int\limits_{\mathtt{S}} d^2x < O^i(x) >_T - \int\limits_{\mathtt{R}^2} d^2x < O^i(x) >_0] \tag{29}$$

If we consider this expression at a nontrivial fixed point[3] λ_*, then at this point the operators O^i have anomalous dimensions $2 - d_i - \gamma_i(\lambda_*)$. At the fixed point the theory is scale invariant, so we expect that the expectation

[3]The trivial fixed point is $\lambda = 0$.

values of the operators with nonzero dimension vanish. The calculations of $< O^i(x) >_T$ and $< O^i(x) >_0$ require the same UV subtractions. Thus as $< O^i(x) >$ have been subtracted in the plane $\mathbf{R^2}$, the expectation value on the strip must vanish at the fixed points [8]. So at the fixed points the function c is stationary

$$\frac{\partial c}{\partial \lambda_i} = 0 \tag{30}$$

In fact we have already proved that the function c is monotonic. The derivatives of c do not change sign between two nearest fixed points. Let us establish that the function c is non-increasing along a RG trajectory. From (29) and the fact that $\Theta(x) = 2\pi\beta_i O^i(x)$ (μ^{d_i} is included in the definition O^i) we obtain

$$-\beta_i \frac{\partial c}{\partial \lambda_i} = -\frac{3}{\pi^2 T^2 V} \int\limits_{\mathrm{S}} d^2x < \Theta >_T \leq 0, \tag{31}$$

where β_i is Gell-Mann-Low functions and Θ is the trace of the stress tensor. This inequality is the consequence of the positivity condition for the stress tensor [9]. We choose the UV subtractions is such a way that $< \mathcal{T}_{\mu\nu} >= 0$ in the plane. The stress tensor is not a primary operator and its expectation value can not be simply obtained from its value in the plane. So $< \mathcal{T}_{\mu\nu} >_T$ on the strip does not vanish and can vanish only at a fixed point.

In a scale invariant (conformally invariant) theory the function c is equal to the central charge c_0 of the corresponding conformal field theory. The finite temperature correction to the free energy density in conformal field theory is given by [3]

$$\mathcal{F}(T) = \mathcal{F}(0) - \frac{\pi c_0}{6} T^2 \tag{32}$$

The free energy density does not acquire an anomalous dimension under renormalization so $c(\lambda, T, a)$ needs no substractions. This fact implies

$$c(\lambda, T, a) = C(g, \frac{\Lambda}{T}) \tag{33}$$

where g is a set of renormalizable coupling constants and Λ is a renormalization mass scale. In (33) $C(g, \Lambda/T)$ is finite when the cut-off is removed keeping g fixed. The dependence on Λ and T has to be as shown in (33) since c is dimensionless. The function C satisfies the Callan-Symanzik equation

$$(\Lambda \frac{\partial}{\partial \Lambda} + \beta_i(g) \frac{\partial}{\partial g_i}) C(g, \frac{\Lambda}{T}) = 0 \tag{34}$$

with Gell-Mann-Low functions

$$\beta_i(g) = \Lambda \frac{\partial g_i}{\partial \Lambda} \tag{35}$$

These equations can be solved as usual to yield $C = C(g(\Lambda/T))$. All facts which have been proved for c are true for C.

Let us collect all the facts we have found. Within the framework of finite temperature 2D QFT we can introduce a positiv function of the renormalizable coupling constants $C(g_1, g_2, ..., g_n)$ which is non-increasing along a RG trajectory and non-decreasing along a temperature trajectory

$$\Lambda \frac{dC}{d\Lambda} = -T \frac{dC}{dT} = -\beta_i(g) \frac{\partial C}{\partial g_i} \ \leq \ 0 \tag{36}$$

and which is stationary only at fixed points

$$\frac{\partial C}{\partial g_i} = 0 \quad \Longleftrightarrow \quad \beta_i(g) = 0 \tag{37}$$

At the critical fixed points, the 2D QFT becomes conformally invariant and the value of C at these points is the same as the corresponding central charge.

The proof presented here is not the only possible one within the framework of the finite size approach. This approach is very rich in possibilities. For example, one can represent the C-function as a two-point function for the stress tensor [10] and study its properties.

Let us consider the connection between the C-theorem at finite temperature and Zamolodchikov's original C-theorem at zero temperature. The naive expectation is that if the temperature goes to zero then our finite temperature C-function will become Zamolodchikov's C-function for 2D QFT at zero temperature. But this is not true. At zero temperature the finite temperature 2D QFT becomes conformally invariant

$$\lim_{T \to 0} C(g(\frac{\Lambda}{T})) = c_{IR} \tag{38}$$

This result can be understood if the finite temperature 2D QFT is interpreted as an infinite one dimesional statistical system. In such a system the phase transition occurs at zero temperature and there are no other phase transitions at finite temperature [11]. At the phase transition point the system has

conformal invariance. So the IR conformal point corresponds to the phase transition at zero temperature.

There is another, more technical, explanation of the result (38). In zero temperature QFT we need an additional mass scale Λ_0 to define the C-function

$$C(g_1, g_2, ..., g_n) = C(g_1(\frac{\Lambda}{\Lambda_0}), g_2(\frac{\Lambda}{\Lambda_0}), ..., g_n(\frac{\Lambda}{\Lambda_0})) \tag{39}$$

This can be seen from simple dimensional analysis. In finite temperature QFT the temperature T plays the role of Λ_0. In a quantum field theory frame we can understand Λ_0 or T as some kind of infrared cut-off. At the conformal points the free enrgy becomes independent of T, which is just an infrared cut-off and can be removed ($T \to 0$). If we introduce into the finite temperature prescriptions an additional mass parameter Λ_0, then we obtain the limit

$$\lim_{T\to 0} C(g(\frac{\Lambda}{\Lambda_0}, \frac{T}{\Lambda_0})) = C(g(\frac{\Lambda}{\Lambda_0})). \tag{40}$$

But in finite temperature QFT the additional parameter Λ_0 does not really play any physical role.

5 Thermodynamical aspects of the C-theorem

Consider a system of interacting particles in equilibrium where the interactions are described by 2D QFT. We know how to define the free energy for this system and can define the other thermodynamical functions as well. We will re-express the finite temperature C-theorem in terms of the thermodynamical functions. Let us define the C-function in terms of the free energy density as in (22). The positivity of the C-function is equivalent to the positivity of the entropy

$$C(T) \geq 0 \quad \Longleftrightarrow \quad -\frac{\partial \mathcal{F}}{\partial T} = \mathcal{S} \geq 0 \tag{41}$$

where $\mathcal{S}$ is the entropy density. Positivity of the entropy is postulated in thermodynamics, but can be derived in statistical mechanics. As we have shown, this fact is related to the positivity of the energy of the ground state.

The monotonic property of the C-function (36) can be expressed in terms of the free energy density

$$T\frac{dC}{dT} \geq 0 \quad \Longleftrightarrow \quad \frac{\partial \mathcal{F}(T)}{\partial T} \leq \frac{2}{T}(\mathcal{F}(T) - \mathcal{F}(0)) \tag{42}$$

where we use Nernst's theorem $\mathcal{S}(0) = 0$. The special inequality (42) for the the free energy density in not a consequence of the thermodynamical laws. If we represent the free energy density in the standard way in terms of the internal energy density $\mathcal{E}$ and the entropy density $\mathcal{S}$

$$\mathcal{F}(T) = \mathcal{E}(T) - T\mathcal{S}(T) \tag{43}$$

then the condition (42) can be rewritten as

$$\mathcal{S}(T) \leq \frac{2}{T}(\mathcal{E}(T) - \mathcal{E}(0)) \tag{44}$$

The inequality (44) can be proved straightforwardly for one dimensional statistical systems and is related to the fact that there are no phase transitions at nonzero temperature, i.e. no long range order for $T > 0$.

In general, a point where a phase transition at finite temperature T occurs is defined to be a point where the free energy F is non-analytic. By non-analytical we mean that the function can not be expanded in a Taylor series around that point. So in one dimensional models the free energy is an analytical function of temperature and can be represented as a Taylor series[4]. Let us use the definition of the specific heat at constant volume

$$C_V(T) = T\frac{\partial \mathcal{S}}{\partial T} = \frac{\partial \mathcal{E}}{\partial T} \tag{45}$$

which can be integrated using Nernst's theorem $\mathcal{S}(0) = 0$

$$\mathcal{S}(T) = \int_0^T \frac{C_V(T')}{T'} dT' \tag{46}$$

[4]In one dimensional models the situation is in fact more sophisticated. We have to evaluate the free energy in every concrete situation. But there exists general theorems (Van Hove's theorems) which show that one dimensional systems with finite range interactions do not exhibit phase transition and the free energy for these systems is an anlytical function of the temperature [12].

$$\mathcal{E}(T) - \mathcal{E}(0) = \int_0^T C_V(T')dT' \tag{47}$$

Using the assumption of analyticity of C_V

$$C_V(T) = \sum_{k=1}^{\infty} C_{Vk} T^k \qquad C_V(0) = 0 \tag{48}$$

we can show that our system satisfies the condition (44). The stationarity of the C-function implies

$$T\frac{dC}{dT} = 0 \quad \Longleftrightarrow \quad T\frac{\partial \mathcal{F}}{\partial T} = 2(\mathcal{F}(T) - \mathcal{F}(0)) \tag{49}$$

We can solve the differential equation of the free energy density and obtain

$$\mathcal{F}(T) = \mathcal{F}(0) + AT^2 \tag{50}$$

where A is a constant of integration. The expression (50) gives the finite temperature correction to the free energy density in conformal field theory.

We have demonstrated that the finite temperature C-theorem can be proved within the thermodynamics of infinite one dimensional statistical systems. The monotonicity of the C-function is related to the absence of long range order for $T > 0$ in these systems.

Let us generalize the definition of the C-function (22) to QFT in an arbitrary number D of dimensions

$$C(T) = \frac{6}{\pi T^D}[\mathcal{F}(0) - \mathcal{F}(T)]. \tag{51}$$

The function C is dimensionless and positive. The monotonic property of C implies the condition

$$\mathcal{S}(T) \leq \frac{D}{(D-1)T}(\mathcal{E}(T) - \mathcal{E}(0)). \tag{52}$$

As we have discussed this condition is not a consequence of the thermodynamical laws. The thermodynamics requires that the entropy is bounded from below but not from above.

In a general situation the condition (52) can be realised only at low temperature where it is possible that the internal energy density $\mathcal{E}$ dominates

$T\mathcal{S}$ in the free energy density, and the free energy density $\mathcal{F}$ may be minimized[5] by minimizing $\mathcal{E}$. At high temperature, the entropy density $\mathcal{S}$ always dominates in the free energy, and the free energy density $\mathcal{F}$ is minimized by maximizing $\mathcal{S}$. So at high temperature the condition (52) breaks down as well. If the macroscopic states (the vacua) of the system obtained by these two procedures are different, then we conclude that at least one phase transition has occurred at some intermediate temperature[6].

In terms of QFT we can say that at high and low temperatures the quantum system has different vacuas. At high temperature the spontaneously broken symmetry is restored. So we can expect a generalization of the C-theorem for trivial systems only, which describe just one phase, for example the free QFT. We can introduce the C-function for an arbitrary QFT and expect a "low temperature" generalization of Zamolodchikov's C-theorem

$$\Lambda\frac{dC}{d\Lambda} = -T\frac{dC}{dT} \leq 0, \quad T < T_c, \;\; \Lambda > \Lambda_c \tag{53}$$

and phase transitions destroy the monotonic behavior of the C-function.

We have to remark that condition (52) was considered also in an article by Castro Neto and Fradkin [13]. The authors consider a system in statistical mechanics which can be described on a lattice with some characteristic lenght a (UV cut-off). They study the possibility to define a monotonic C-function in the limit $a \to 0$ and find (52) to be a sufficient condition for this. We hope that the treatment presented here within the framework of finite temperature QFT clarifies the origin of the condition (52).

6 The Källen-Lehmann spectral representation at finite temperature

In QFT the basic idea of a spectral representation consists in constraining the form of a two-point correlator by enforcing Poincaré invariance of the propagating intermediate states [14]. The propagation of states with an intermediate mass μ is given by the propagator of a free scalar particle of the same mass. We are going to derive the Källen-Lehmann representation for finite temperature 2D QFT.

[5]In QFT by minimizing the free energy we can find the ground states (vacua).
[6]This argument is usually called the energy-entropy argument.

The first step in this direction is to find a basis of the Hilbert space of the theory. We consider euclidian QFT defined on $\mathbf{R} \times \mathbf{S^1}$, where the radius of $\mathbf{S^1}$ (the inverse temperature) is the parameter of the theory. The Hilbert space consists of eigenfunctions of the Laplace operator $\triangle$

$$\triangle\, |p_n, \mu >= -\mu^2 |p_n, \mu >, \quad p_n^2 = -\mu^2 - (2\pi n T)^2 \tag{54}$$

where the time-like component of the momentum is discrete $p^\nu = (2\pi n T, p)$. These eigenfunctions form a complete basis

$$\Psi_{p_n,\mu}(x_1, x_2) =< x|p_n, \mu >= \frac{e^{ix_2 p_n + x_1\sqrt{\mu^2+p_n^2}}}{(2\sqrt{p_n^2+\mu^2})^{1/2}}. \tag{55}$$

$$\Psi_{p_n,\mu}(x_1 + k\beta, x_2) = \Psi_{p_n,\mu}(x_1, x_2), \quad k \in Z \tag{56}$$

The projection operator on the space of representations of squared mass μ^2 is built as

$$\wp_{\mu^2}(T) = \sum_{n=-\infty}^{\infty} |p_n, \mu >< p_n, \mu| \tag{57}$$

where the projector $\wp_{\mu^2}(T)$ is a function of temperature. An important remark is that for different temperatures we find different Hilbert spaces. The sum over all the representations of the Poincaré group gives the identity operator in the Hilbert space with fixed temperature

$$I = \int d\mu^2 \wp_{\mu^2}(T) \tag{58}$$

Let us calculate the propagator of a free scalar particle of mass μ at temperature T. This propagator can be obtained by inserting the projector $\wp_{\mu^2}(T)$ into the correlation function

$$G(x_1, x_2, \mu, T) \equiv< \phi(x_1, x_2)\phi(0,0) >=< \phi(x_1, x_2)\wp_{\mu^2}(T)\phi(0,0) >=$$
$$= T \sum_{n=-\infty}^{\infty} \int \frac{dp}{(2\pi)} \frac{e^{ipx_2} e^{i2\pi n T x_1}}{p^2 + (2\pi n T)^2 + \mu^2} \tag{59}$$

where we use the normalization $< p_n, \mu|\phi(x_1, x_2)|0 >= \Psi_{p_n,\mu}(x_1, x_2)$.

Now let us consider the correlator of interacting scalar fields at temperature T

$$< S(x_1, x_2)S(0,0) >_T = \int d\mu^2 < S(x_1, x_2)\wp_{\mu^2}(T)S(0,0) >_T \tag{60}$$

The amplitudes $< S(x_1,x_2)|p_n,\mu >$ and $< \phi(x_1,x_2)|p_n,\mu >$ transform in the same way under the action of the Poincaré group and have the same quantum numbers. Therefore they are equal up to a normalization, which can only depend on physical parameters of our theory (the temperature and the Casimirs of the groups of symmetries). In our case we have

$$< S(x_1,x_2)|p_n,\mu >_T = N_S(\mu^2,T) < \phi(x_1,x_2)|p_n,\mu >_T \tag{61}$$

From (60) and (61) we can obtain the spectral representation of two-point Green function at the temperature T

$$< S(x_1,x_2)S(0,0) > = \int d\mu^2 c_S(\mu^2,T) G(x_1,x_2,\mu,T) \tag{62}$$

$$c_S(\mu^2,T) = N_S^2(\mu^2,T)$$

where the function $c_S(\mu^2,T)$ is called the spectral density.

7 The C-theorem and spectral representation at finite temperature

Let us consider the Källen-Lehmann spectral representation of the correlator of two stress tensor on the strip $\mathbf{R}\times[0,\beta]$,

$$< \mathcal{T}_{\mu\nu}(x)\mathcal{T}_{\rho\sigma}(0) >_T = \frac{\pi}{3}\int_0^\infty d\mu c(\mu,T) \sum_{n=-\infty}^{\infty} T$$

$$\times \int \frac{dp}{2\pi} e^{ipx_2} e^{i2\pi nTx_1} \frac{(g_{\mu\nu}(p^2+(2\pi nT)^2) - p_\mu p_\nu)(g_{\rho\sigma}(p^2+(2\pi nT)^2) - p_\rho p_\sigma)}{p^2+(2\pi nT)^2+\mu^2} \tag{63}$$

where the time-like component of the momentum is $p_1 = 2\pi nT$ and the space-like component is $p_2 \equiv p$. As we have discussed, in two dimensions there exists only one Lorentz invariant object which is allowed in the spectral representation. Therefore we have only one unknown scalar function namely the spectral density $c(\mu,T,\Lambda)$ which depends on the intermediate mass scale μ, a mass scale Λ of the theory and the temperature T, as well as on dimensionless coupling constants. Unitarity of the theory implies that

$c(\mu, T, \Lambda) \geq 0$. The function $c(\mu, T)d\mu$ is dimensionless. In a scale invariant theory the form of $c(\mu, T)$ is fixed by its dimesionality

$$c(\mu, T) \sim \delta(\mu) \qquad c(\mu, T) \sim \delta(T) \tag{64}$$

$$c(\mu, T) \sim \frac{1}{\mu} \qquad c(\mu, T) \sim \frac{1}{T} \tag{65}$$

The third and fourth possibilities cause IR problems at $\mu = 0$ or at $T = 0$. So in a scale invariant theory the spectral density is

$$c(\mu, T) = c_0\delta(\mu) + c_0'\delta(T) \tag{66}$$

When we substitude (66) in the spectral representation for the two-point function of trace of the stress tensor

$$< \Theta(x)\Theta(0) >= 0 \tag{67}$$

then the trace annihilates the vacuum and the theory is conformally invariant. The general form of spectral density is

$$c(\mu, T) = c_0\delta(\mu) + c_0'\delta(T) + \frac{1}{T}c_1(\frac{\mu}{T}, \frac{\Lambda}{T}) \tag{68}$$

where the last term can be represented as shown in (68). Therefore the spectral density can not develop an anomalous dimension so the behavior under RG flow is given by dimensional analysis. The last term describes the behavior of spectral density away from the critical points $\mu = 0$ and $T = 0$.

The form of (68) is the only possible one for spectral density $c(\mu, T, \Lambda)$ in 2D QFT at finite temperature. Considering the limit

$$\lim_{\lambda \to 0} \lambda \, d\mu \; c(\lambda\mu, \lambda T, \Lambda) = \lim_{\lambda \to 0} d\mu \; c(\mu, T, \frac{\Lambda}{\lambda}) \tag{69}$$

we see that the points at $\mu = 0$ and at $T = 0$ describe the IR fixed point of the theory ($\Lambda \to \infty$). In the IR limit the massive degrees of freedom effectively become massless and the spectral density is compressed to delta functions. The IR limit of 2D QFT is described by a conformal field theory.

The limit

$$\lim_{\lambda \to \infty} \lambda \, d\mu \; c(\lambda\mu, \lambda T, \Lambda) = \lim_{\lambda \to \infty} d\mu \; c(\mu, T, \frac{\Lambda}{\lambda}) \tag{70}$$

gives the UV fixed point of the theory ($\Lambda \to 0$). The short (UV) distance behavior of the two-point function (63) is

$$< \mathcal{T}_{zz}(z, \bar{z}) \mathcal{T}_{zz}(0,0) > \sim \frac{1}{z^4}, \qquad z \to 0 \tag{71}$$

so the UV fixed point corresponds to a conformal field theory.

The C-function can be introduced by integrating the smooth part of the spectral density with smearing functions $f(x)$ which has following properties: $f > 0$, $f(0) = 1$, $f(x)$ decreases expotentially as $\mu \to \infty$ and $xdf/dx \leq 0$

$$C(g(\frac{\Lambda}{T})) = c_{IR} + \int \frac{d\mu}{T} c_1(\frac{\mu}{T}, \frac{\Lambda}{T}) f(\frac{\mu}{T}) = c_{IR} + \int dx c_1(x, \frac{\Lambda}{T}) f(x) \tag{72}$$

Then the derivative along RG flow is

$$-\beta_i(g) \frac{\partial}{\partial g_i} C = \Lambda \frac{\partial}{\partial \Lambda} C = \int dx c_1(x, \frac{\Lambda}{T}) x \frac{d}{dx} f(x) \leq 0 \tag{73}$$

where we have used dimensional arguments and integrated by parts. The other technical details of the proof can be generalized in a straightforward manner from the standard proof of the zero temperature C-theorem [2].

8 Conclusions

We have considered an equilibrium system of particles which are described by 2D QFT. This system can be studied within the framework of QFT at finite temperature. We have proved that for these system there exists a function $C(g_1, g_2, ..., g_n)$ of the coupling constants which is non-increasing along renormalization group trajectories and non-decreasing along temperature trajectory and stationary only at the fixed points. This finite temperature generalization of Zamolodchikov's C-theorem gives us the solid ground for treating quantum theory in the context of thermodynamics. We have shown that the C-theorem is not a consequence of the thermodynamical laws and is based on a special assumption about analyticity of the free energy. This assumption is fulfilled only in one dimensional statistical systems, where there is no long range order for finite temperature $T > 0$. In one dimensional statistical systems the phase transition occurs only at zero temperature, which is described by an IR conformal theory.

We have presented three proofs of the finite temperature C-theorem. The first proof utilizes the finite size approach, which is widely used for studying of two dimensional field theories. This approach is very rich in possibilities and is related to concrete calculations. The definition of the C-function depends on the choice of renormalization prescriptions. The second proof is presented in the context of the thermodynamics of one dimensional statistical systems. The monotonicity of the C-function implies a special condition for the entropy and internal energy, which can be shown to hold under the assumption of analyticity of the specific heat. The thermodynamical proof of the C-theorem teaches us that the monotonic behavior of the C-function is not universal and can be expected only at low temperature (low energy) in arbitrary non-two dimensioanl QFT. A phase transition destroys the monotonicity of the C-function. The third proof uses the finite temperature generalization of the Källen-Lehmann spectral representation for the two-point function of the stress tensor. In the two dimensonal case there is only one Lorentz invariant which is allowed in the spectral representation. The absence of anomalous dimension for the spectral density means that we can use dimensonal analysis. The dimesionality of $c(\mu, T)$ completely determines the form and properties of spectral density.

The first and third proofs show no trivial relation between the finite temperature C-theorem and Zamolodchikov's C-theorem at zero temperature. We have a phase transition at zero temperature and the theory has conformal invariance at this point. From the point of view of QFT at finite temperature the system becomes scale invariant at zero temperature.

There are a lot of questions left. We have to understand the role of spontaneous symmetry breaking in two dimensioanl QFT. The finite temperature C-theorem tells us that there are no phase transitions at temperatures $T > 0$. This means that the broken symmetry can be restored only at zero temperature, so in two dimensional QFT no spontaneous symmetry breaking can occur at all. If this is true, then we have to prove this in a rigorous way. The famous theorem of Coleman [15] gives us some insight into these problems. Using the argument, that the scalar field in 2D QFT is dimensionless, Coleman has proved that there is no Goldstone effect in two dimensions.

Another problem is related to the generalization of Zamolodchikov's C-theorem to QFT in an arbitrary number of dimensions. The thermodynamical arguments show that it is impossible to define a monotonic function of temperature starting from the free energy (or partitition function).

This simply means that there is no naive or straightforward generalization of Zamolodchikov's theorem. Zamolodchokov's theorem can be generalized as a criteria of the existence of different phases in the quantum system.

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